

STABILITY OF MULTIVALUED ATTRACTORS

MIROSLAV RYPKA

Dept. of Math. Anal. and Appl. of Math., Faculty of Science,
Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic
e-mail: miroslav.rypka01@upol.cz

Abstract.

Stimulated by recent problems in the theory of iterated function systems, we provide a variant of the Banach converse theorem for multivalued maps. In particular, we show that attractors of continuous multivalued maps in a metric space are stable. Moreover, such attractors in locally compact, complete metric spaces may be obtained by means of the Banach theorem in the hyperspace.

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1. Introduction

Multivalued maps and their attractors are studied in relation to dynamical systems, e.g. iterated function systems or differential inclusions. Throughout the whole paper, we consider continuous multivalued maps with compact values which generate continuous operators on hyperspaces, as discussed in the next section.

Our motivation is following. We would like to state a variant of Jánoš theorem for operators on hyperspaces induced by multivalued maps. Under Jánoš theorem we understand the results on the converse of Banach theorem developed in [Ja], [Me], [Jach], [Le1], [Le2], [Op]. In spite of the metric nature of the Banach theorem, these papers provide several topological conditions on a map to be contractive.

Although, the theory of the converse to the Banach theorem seems complete, analogical problems in the theory of multivalued maps (for detailed treatment of attractors of multivalued maps see e.g. [AF], [AFGL], [Na]) and iterated function systems are still addressed. Since the Hutchinson's seminal work [Hu] (see also [Wi]), the metric approach to attractors of IFSs dominated. With only a few exceptions ([Ki], [LM1], [LM2], [LM3]), the attractors of IFS were obtained by means of the Banach theorem. Recently, it was pointed out [ABVW], [BLR1], [BLR2] that the attractor of IFS is a topological notion and

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the contractivity of maps in an IFS is only a sufficient condition for the existence of an attractor (for different approaches see e.g. [BN], [Mi]). Novelty of this fact is caused by the prevailing interest in affine IFSs in Euclidean spaces, for which the existence of an attractor is equivalent to the existence of equivalent metric in original space w.r.t. which the maps in IFS are contractions [ABVW].

The question whether it holds for any IFS with point fibred attractor A was raised by Kameyama ([Ka]). *Does there exist a metric on A such that Hutchinson operator $F|_A$ is contractive and the topology on A induced by this metric is the same as the topology of X restricted to A ?*

Similar problem for multivalued maps was stated by Fryszkowski ([Jach]). *Let X be an arbitrary nonempty set, 2^X be the family of all nonempty subsets of X and $F : X \rightarrow 2^X$ be a multivalued map. Find necessary conditions and (or) sufficient conditions for the existence of a complete metric d for X such that given $c \in (0, 1)$, F would be a Nadler ([Na]) multivalued c -contraction with respect to d , that is*

$$d_H(F(x), F(y)) \leq cd(x, y) \forall x, y \in X,$$

where d_H denotes the Hausdorff metric generated by d .

In contrast to Fryszkowski problem or Kameyama question, we shift the search for the metric w.r.t. which a map is contracting to the hyperspace. In particular, we will explore the operator in hyperspace induced by multivalued map. We will proceed in the following way. Next section recalls the basic notions, e.g. multivalued maps, attractors and strict attractors. Main results can be found in Section 3. In Theorem 1 we prove that attractors of multivalued maps are stable fixed points of associated operators in hyperspace. The stability of the attractor is implied by the monotonicity of such operators. Corollary 2 provides a variant of Jánoš theorem for operators generated by multivalued maps in locally compact, complete metric spaces. The same conditions as in Theorem 1 imply also the stability of strict attractors. Hence, we express the analogical results for strict attractors in Corollary 1 and 3. Finally, a few examples are provided. Example 2 shows that we cannot drop monotonicity condition. Operators in a hyperspace need not be generated by multivalued maps. Attractivity of multivalued operators does not imply their contractivity even in compact spaces. Example 3 illustrates relation of our theory to Fryszkowski problem. It proves that multivalued maps generating contractive operators need not be contractions, even in compact metric spaces.

2. Notation

Throughout the whole paper, we deal with a metric space (X, d) . Let us denote by $\mathbb{K}(X)$ the space of compact subsets of X , called the *hyperspace*, endowed usually with the Hausdorff metric d_H defined (cf.

e.g. [Hu])

$$d_H(A, B) := \inf\{r > 0; A \subset O_r(B) \text{ and } B \subset O_r(A)\},$$

where $O_r(A) := \{x \in X; \exists a \in A : d(x, a) < r\}$ and $A, B \in \mathbb{K}(X)$.
An alternative definition reads

$$\begin{aligned} d_H(A, B) &:= \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\} \\ &= \max\{\sup_{a \in A} (\inf_{b \in B} d(a, b)), \sup_{b \in B} (\inf_{a \in A} d(a, b))\}. \end{aligned}$$

REMARK 1. The Hausdorff metric is induced by d . However, there exist metrics in $\mathbb{K}(X)$ which cannot be induced by any d .

Thus, we will denote a general metric in $\mathbb{K}(X)$ by D .

We will often employ a neighbourhood of a compact set as a point in a hyperspace.

DEFINITION 1. Let $A \in (\mathbb{K}(X), d_H)$. We will write $\mathcal{N}_r(A) := \{B \in \mathbb{K}(X); d_H(A, B) < r\}$

In general, letters such as $\mathcal{N}, \mathcal{O}, \mathcal{B} \dots$ will stand for classes of sets.

DEFINITION 2. The map $F : X \rightarrow \mathbb{K}(X)$ is called *multivalued map* and the operator $F : \mathbb{K}(X) \rightarrow \mathbb{K}(X)$, defined by

$$F(A) = \bigcup_{x \in A} F(x),$$

is called *multivalued operator*.

In the paper, all the multivalued maps and operators are continuous w.r.t. d and d_H . Observe that continuous multivalued map on a metric space generates continuous multivalued operator (cf. [AF]).

DEFINITION 3. Let (X, d) be a metric space. A map $f : X \rightarrow X$ is a contraction if for some $c \in [0, 1)$, $d(f(x), f(y)) \leq cd(x, y)$ for any $x, y \in X$.

Multivalued maps and operators are often generated by iterated function systems (IFSs).

DEFINITION 4. An *Iterated function system* consists of finite number of continuous maps f_i , $i = 1, 2, \dots, N$, on a metric space (X, d) .

Any IFS yields a multivalued map $F : X \rightarrow \mathbb{K}(X)$,

$$F(x) = \bigcup_{i=1}^N f_i(x),$$

and a multivalued operator $F : \mathbb{K}(X) \rightarrow \mathbb{K}(X)$,

$$F(A) = \bigcup_{x \in A} F(x),$$

called *Hutchinson operator*.

Usually, IFSs of contractive maps f_i on complete metric spaces are treated. They possess attractor $A^* \in \mathbb{K}(X)$ due to the Banach theorem.

Notice that this theorem gives not only existence, but also attractivity and numerical stability necessary for visualization of the attractor.

Let us discuss the notion of attractor.

DEFINITION 5. Let (X, d) be a metric space and $f : X \rightarrow X$ continuous with a fixed point x^* . The point x^* is called *attractive* if for all $x \in X$,

$$\lim_{n \rightarrow \infty} d(f^n(x), x^*) = 0.$$

DEFINITION 6. Let (X, d) be a metric space and $f : X \rightarrow X$ continuous with a fixed point $x^* \in X$. The point x^* is called *stable* if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(f^n(x), x^*) < \epsilon \forall n \in \mathbb{N}, x \in X, d(x, x^*) < \delta.$$

The fixed point x^* is *asymptotically stable* if it is attractive and stable.

DEFINITION 7. Let (X, d) be a metric space and $F : X \rightarrow K(X)$ be a continuous multivalued map. Let $A^* \in K(X)$ be such that $F(A^*) = A^*$ and $U \subset K(X)$ open be such that $\lim_{n \rightarrow \infty} d_H(F^n(B), A^*) = 0 \forall B \in U$. Then A^* is called *an attractor of multivalued map F* .

One possible definition of an attractor of IFS employs the previous definition.

DEFINITION 8. Let $\{X; f_1, f_2, \dots, f_N\}$ be an IFS and F its Hutchinson operator with a fixed point $A^* \in K(X)$. A^* is attractive if there exists $\mathcal{U} \subset K(X)$ open such that for all $B \in \mathcal{U}$

$$F^n(B) \rightarrow A^*.$$

However, this definition has its drawbacks in hyperspaces, as can be seen from the following example.

EXAMPLE 1. The IFS $\{([-1, 1], d_{eucl}); f(x) = \sqrt[3]{x}\}$ possesses three overlapping attractors

$$\begin{aligned} A_1 &= \{-1\}, U_1 = K([-1, 0)) \\ A_2 &= \{1\}, U_2 = K((0, 1]) \\ A_3 &= \{-1, 1\}, U_3 = K([-1, 1] \setminus \{0\}) \setminus (U_1 \cup U_2) \end{aligned}$$

In order the attractors do not overlap, we introduce the notion of strict attractor (see [BLR1], [BLR2]).

DEFINITION 9. A compact set $A^* \subset X$ is a *strict attractor* of F , if there exists an open set $U \supset A^*$ such that

$$\forall S \in K(U), F^n(S) \rightarrow A^*.$$

The maximal open set U with the above property is called the *basin of attraction* of the attractor A^* (with respect to F) and denoted by $B(A^*)$.

REMARK 2. The existence of the maximal open set U is proven in [BLR2].

REMARK 3. Strict attractor is topological invariant ([BHR, Lemma 2.8]) and it is an attractor.

3. Results

THEOREM 1. *Any attractor of continuous multivalued map on a metric space (X, d) is asymptotically stable in $(\mathbb{K}(X), d_H)$.*

The proof of the theorem proceeds in two steps. First, we show stability in $\mathbb{K}(A^*)$. Then we extend it to the whole hyperspace $\mathbb{K}(X)$.

LEMMA 1. *Let (A^*, d) be a compact metric space. Let $F : A^* \rightarrow \mathbb{K}(A^*)$ be such that $F(A^*) = A^*$ and*

$$\exists \epsilon > 0 : \lim_{n \rightarrow \infty} d_H(F^n(\{B\}), A^*) = 0, \forall B \in \mathcal{N}_{\epsilon_{A^*}}(A^*).$$

Then F is asymptotically stable.

PROOF. We only need to show the stability of A^* . If A^* is a singleton, then it is obviously stable in $\mathbb{K}(A^*)$.

Suppose that A^* is not a singleton, which means $\text{diam}(A^*) > 0$. We will proceed by contradiction. Assume A^* is not stable. Then there exist $\epsilon > 0$ and a sequence

$$\{B_n\} \in \mathbb{K}(A^*), d_H(B_n, A^*) < \min \left\{ \frac{1}{n}, \epsilon_{A^*} \right\}$$

such that

$$\exists k_n \in \mathbb{N} : d_H(F^{k_n}(B_n), A^*) > \epsilon.$$

Observe that $k_n \rightarrow \infty$, otherwise the operator F would not be continuous.

Since F is continuous, for any $n \in \mathbb{N}$, there is an open set $\mathcal{B}_n \subset \mathbb{K}(A^*)$, such that

$$\mathcal{B}_n = \{B \in \mathcal{N}_{\epsilon_{A^*}}(A^*); d_H(F^{k_n}(B), A^*) > \epsilon\}.$$

In the following part, we will employ the monotonicity of F . Notice that for any set $C \in \mathbb{K}(A^*)$, $C \subset B \in \mathcal{B}_n$

$$d_H(F^{k_n}(C), A^*) > \epsilon \tag{1}$$

since

$$d_H(F^{k_n}(B), A^*) \leq d_H(F^{k_n}(C), A^*).$$

We will show that there exists a set $K \in \mathbb{K}(A^*)$ such that K belongs to the subsequence $\overline{\mathcal{B}_n}$. Denote by $\tilde{\mathcal{B}}$ the set $\{C \in \mathcal{N}_{\epsilon_{A^*}}(A^*); \exists B \in \mathcal{B}, B \subset C\}$. The set $\tilde{\mathcal{B}}$ is open for $\mathcal{B} \in \mathbb{K}(A^*)$ open. Furthermore, let $(\mathcal{B}' \cap \tilde{\mathcal{B}})^{-1}$ stand for the set $\{B \in \mathcal{B}; \exists B' \in \mathcal{B}', B \subset B'\}$ which is again open for $\mathcal{B}, \mathcal{B}'$ open.

Observe that for any $n_1 \in \mathbb{N}$ there exists $n_2 \in \mathbb{N}$ such that $\mathcal{B}_{n_1 n_2} := (\mathcal{B}'_{n_2} \cap \widetilde{\mathcal{B}_{n_1}})^{-1}$ is nonempty and open. Since \mathcal{B}_{n_1} is nonempty and open, there exists an open ball $\mathcal{O}_{n_1} \subset \mathcal{B}_{n_1}$ with radius r_{n_1} . For $n_2 \in \mathbb{N}$, $n_2 > \frac{1}{r_{n_1}}$, consider the set \mathcal{B}_{n_2} . The inequality $d_H(B_{n_2}, A^*) < r_{n_1}$ and (1) imply that $(\mathcal{B}_{n_2} \cap \widetilde{\mathcal{B}_{n_1}})^{-1}$ is nonempty. It is also open, since \mathcal{B}_{n_1} and

\mathcal{B}_{n_2} are open. We will simplify the notation using $\mathcal{B}_{n_1 n_2}$ instead of $(\mathcal{B}_{n_2} \cap \widetilde{\mathcal{B}_{n_1}})^{-1}$.

Again, since $\mathcal{B}_{n_1 n_2}$ is open, there exists an open ball $\mathcal{O}_{n_1 n_2} \subset \mathcal{B}_{n_1 n_2}$ with radius $r_{n_1 n_2}$ and $n_3 \in \mathbb{N}$ such that

$$\mathcal{B}_{n_1 n_2 n_3} := (\mathcal{B}_{n_3} \cap \widetilde{\mathcal{B}_{n_1 n_2}})^{-1}$$

is nonempty and open. Repeating this process to infinity, we obtain the sequence

$$\mathcal{B}_{n_1}, \mathcal{B}_{n_1 n_2}, \mathcal{B}_{n_1 n_2 n_3}, \mathcal{B}_{n_1, n_2, n_3 n_4}, \dots \quad (2)$$

Observe that the sequence is nested, i.e.

$$\mathcal{B}_{n_1} \supset \mathcal{B}_{n_1 n_2} \supset \mathcal{B}_{n_1 n_2 n_3} \supset \mathcal{B}_{n_1 n_2 n_3 n_4} \supset \dots$$

and for any $p \in \mathbb{N}$,

$$d_H(F^{n_p}(B), A^*) > \epsilon, \quad B \in \mathcal{B}_{n_1 n_2 \dots n_p}.$$

Last, consider the sequence of closures

$$\overline{\mathcal{B}_{n_1}}, \overline{\mathcal{B}_{n_1 n_2}}, \overline{\mathcal{B}_{n_1 n_2 n_3}}, \overline{\mathcal{B}_{n_1 n_2 n_3 n_4}}, \dots \quad (3)$$

where $\overline{\mathcal{B}_{n_1 \dots n_k}}$ is compact in $\mathbb{K}(A^*)$, for any k . Since the sequence (3) is also nested, there is a nonempty intersection $\overline{\mathcal{B}} = \bigcap_{i=1}^{\infty} \overline{\mathcal{B}_{n_1 n_2 \dots n_i}}$. Continuity of F implies

$$d_H(F^{n_i}(K), A^*) \geq \epsilon, \quad i \in \mathbb{N}, \quad K \in \overline{\mathcal{B}},$$

which is a contradiction to the attractivity of A^* in $\mathcal{N}_{\epsilon_{A^*}}$. \square

PROOF. (Continuation of proof of Theorem 1) Let us proceed to the second part of the proof, where we will use uniform stability of A^* in a compact subset of $\mathcal{N}_{\epsilon_A^*}$ and uniform continuity of F in feasible compact subset of $\mathbb{K}(X)$. Assume that A^* is not stable. Then there exists $\epsilon > 0$, a sequence of sets $B_n \in \mathcal{U}$, $d_H(B_n, A^*) < \frac{1}{n}$ and a sequence $i_n \in \mathbb{N}$ such that

$$d_H(F^{i_n}(B_n), A^*) > \epsilon, \quad \forall n \in \mathbb{N}.$$

For the sake of simplicity, let us denote $F^{i_n}(B_n)$ by B_n^i .

Observe that

$$\overline{\{B_n\}} = \bigcup_{n=1}^{\infty} B_n \cup A^*$$

is a compact subset of X as well as

$$\hat{B} := \bigcup_{i=1}^{\infty} F^i(\overline{\{B_n\}}).$$

The compactness of \hat{B} implies that $\forall \epsilon \exists m_0 \in \mathbb{N}$ such that

$$d_H(F^i(\hat{B}), A^*) < \epsilon \quad \forall i \geq i_0.$$

Similarly, from the monotonicity of F , we have

$$\forall \epsilon \exists i_0 \in \mathbb{N} \forall i \geq i_0 \forall B_n^i \exists C_n^i \in K(A^*) \cap \mathcal{U} : d_H(B_n^i, C_n^i) < \epsilon.$$

Since F is stable on $\mathcal{N}_{\epsilon_{A^*}}$, it is also uniformly stable on any of its compact subsets. Consider closed neighbourhood $\mathcal{U} \in \mathbb{K}(A^*)$ such that $\mathcal{U} \subset \mathcal{N}_{\epsilon_{A^*}}$.

Without loss of generality, let $\epsilon > 0$ be such that $\mathcal{N}_\epsilon(A^*) \subset \mathcal{U}$. The uniform stability of F implies

$$\forall \epsilon > 0 \exists k_0 \in \mathbb{N} : d_H(F^k(B), A^*) < \frac{\epsilon}{2}, k \geq k_0, \forall B \in \mathcal{U}. \quad (4)$$

The operator F is uniformly continuous on any compact subset of $\mathbb{K}(X)$, which implies $\forall k \in \mathbb{N} \forall \epsilon > 0 \exists \delta_0 > 0$:

$$d_H(F^k(B), F^k(B')) < \frac{\epsilon}{2}, \forall B, B' \in \mathbb{K}(\hat{B}), d_H(B, B') < \delta_0. \quad (5)$$

Let k_0 fulfill (4) and δ_0 fulfill (5) for δ_0 . From attractivity of F and from $F^n(\hat{B}) \rightarrow 0$, we get

$$\forall \delta > 0 \exists n_0 \in \mathbb{N} \forall i \in \mathbb{N} \forall B_n^i \exists C_n^i \in \mathbb{K}(A) : d_H(B_n^i, C_n^i) < \delta. \quad (6)$$

Notice that there always exists $C_n^i \in \mathbb{K}(A)$

$$d_H(C_n^i, A^*) \leq d_H(B_n^i, A^*). \quad (7)$$

Consider a sequence $\{i_n\} \in \mathbb{N}$ such that

$$d_H(B_n^{i_n}, A^*) \geq \epsilon \quad (8)$$

and

$$d_H(B_n^i, A^*) \geq \epsilon \Rightarrow i > i_n.$$

Observe that $i_n \rightarrow \infty$, otherwise F would not be continuous.

Let us investigate the behaviour of F^{k_0} on $B_n^{i_n - k_0}$ for $n \in \mathbb{N}$ such that $i_n > k_0$. From (4) and (5), we obtain (notice that $d_H(C_n^{i_n - k_0}, A^*) < \epsilon$ is implied by (3) and (7))

$$d_H(F^{k_0}(B_n^{i_n - k_0}), A^*) \leq d_H(F^{k_0}(B_n^{i_n - k_0}), C_n^{i_n - k_0}) + d_H(F^{k_0}(C_n^{i_n - k_0}), A^*)$$

implying

$$d_H(B_n^{i_n}, A^*) < \epsilon$$

which is a contradiction to (8). \square

Since a strict attractor is an attractor, we obtain:

COROLLARY 1. *Any strict attractor of continuous multivalued map on a metric space (X, d) is asymptotically stable in $(\mathbb{K}(X), d_H)$.*

Adding conditions of completeness and local compactness on an original space X , attractive multivalued induce a contraction in hyperspace. Notice that a hyperspace $(\mathbb{K}(X), d_H)$ inherits most of the features of a metric space (X, d) .

REMARK 4. (cf. e.g. [AR]) Let (X, d) be a metric space. The hyperspace $(\mathbb{K}(X), d_H)$ is complete if and only if (X, d) is complete. The hyperspace $(\mathbb{K}(X), d_H)$ is locally compact if and only if (X, d) is locally compact.

LEMMA 2. (cf. e.g. [Op, Theorem 2.1]) *Let (X, d) be a locally compact, complete metric space. Let $f : X \rightarrow X$ be a continuous map with a fixed point $x^* \in X$ such that x^* is attractive and stable. Then there exists metric d' equivalent to d such that f is a contraction in (X, d') .*

Theorem 1 and Lemma 2 imply a corollary.

COROLLARY 2. *Let (X, d) be a locally compact, complete metric space. Let $F : X \rightarrow \mathbb{K}(X)$ be a multivalued map with an attractor A^* and basin of attraction \mathcal{U} . There exists metric D in $\mathbb{K}(X)$ equivalent with d_h , such that the operator $F : \mathbb{K}(X) \rightarrow \mathbb{K}(X)$ is a contraction in \mathcal{U} .*

We immediately receive the following.

COROLLARY 3. *Let (X, d) be a locally compact, complete metric space. Let $F : X \rightarrow \mathbb{K}(X)$ be a multivalued map with a strict attractor A^* and basin of attraction $B(A)$. There exists metric D in $\mathbb{K}(X)$ equivalent with d_H , such that the operator $F : \mathbb{K}(X) \rightarrow \mathbb{K}(X)$ is a contraction in $\mathbb{K}(B(A))$.*

The metric D may be constructed by means of [Op] or [Ja]. However, it need not be, in general, induced by any d on X . In general, stable attractors in compact metric spaces do not fulfill Fryszkowski condition.

EXAMPLE 2. Consider the following IFS. $F = \{(X, d), f_1, f_2\}$, $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, where d is ordinary Euclidean metric and $f_1(x, y) := (x, y)$, $f_2(x, y) := (x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha)$, where α/π is irrational. We will show that there is no metric d' on X equivalent to d such that multivalued map $F : X \rightarrow \mathbb{K}(X)$ associated to the IFS is a contraction (w.r.t. d' and d'_H). We will prove it by contradiction.

Suppose, there exists such metric d' . Then we can find a point $x_0 \in X$ and $\epsilon > 0$ such that $d'(x, x') \leq d(f_2(x), f_2(x'))$, $x, x' \in N_\epsilon(x_0)$. Otherwise, f_2 would possess periodic points according to [Ed, 4 Theorem 2] and d' would not be equivalent to d .

Since $0 < \alpha < 2\pi$, we can choose $x, x' \in N_\epsilon(x_0)$ close enough so that

$$d(f_2(x), \{x, x'\}) \geq d(x, x') \text{ and } d(f_2(x'), \{x, x'\}) \geq d(x, x').$$

Then

$$d'_H(F(x), F(x')) = d'_H(\{x, f_2(x)\}, \{x', f_2(x')\}) = d'(f_2(x), f_2(x')) \geq d'(x, x'),$$

which is a contradiction to contractivity of F w.r.t. d' and d'_H .

REMARK 5. For $X = \mathbb{R}^2$, the IFS does not possess an attractor.

We can not drop the condition that the operator is generated by a multivalued map. In general, an attracting operator in compact metric space is not a contraction.

EXAMPLE 3. Let us denote by $\mathbb{K}_C(I)$ the set of closed subintervals of $I = [0, 1]$, endowed with Euclidean metric. The space $(\mathbb{K}_C(I), d_H)$ is equivalent to a filled triangle $T = ABC$, $A = [0, 1]$, $B = [1, 1]$, $C = [0, 0]$ in \mathbb{R}^2 endowed with the maximum metric (cf. e.g. [AR2]).

Inspired with an example in [GKLOP, Theorem 10] and [Op, Theorem 2.1], we can construct an operator in $\mathbb{K}(I)$ which is attractive, but not contractive. We consider the space T as a union of triangles T_α , $\alpha \in [0, 1]$ with empty interiors and common point A illustrated in Figure 2. Observe that there is a homotopy $h : T \times (0, 1) \rightarrow T$ such that $T_\alpha = h(\alpha, T_1)$.

Similarly, as discussed in [BLR1], Consider the set of points X of the circle which may be projected to real numbers and infinity on the real line \mathbb{R}^* . Let $f^* : \mathbb{R}^* \rightarrow \mathbb{R}^*$ be such that $f^*(x) = x + 1$, $x \neq \infty$ and $f^*(\infty) = \infty$.

Observe that the induced map $f : X \rightarrow X$ is continuous with respect to the Euclidean metric on the circle. It is obvious that each point of X is attracted to the point ∞ (see Figure 1).

We can construct a homeomorphism $g : X \rightarrow T_1$ such that $A = \infty$. Define a map $G : T \rightarrow T$ such that $G(x) = h(\alpha) \circ g \circ h^{-1}(\alpha)$ for $x \in T_\alpha$ and $G(A) = A$. G is obviously continuous (see Figure 1) and may be applied on $\mathbb{K}_C(I)$.

Let us extend the map $G : \mathbb{K}_C(I) \rightarrow \mathbb{K}_C(I)$ to the whole hyperspace $\mathbb{K}(I)$. Let us define a map $Q : \mathbb{K}(I) \rightarrow \mathbb{K}(I)$,

$$Q(D) = [\min(D), \max D],$$

which is continuous (even retraction).

Defining $F = G \circ Q$ we obtain an operator in $\mathbb{K}(I)$ which has an attractor I . However, operator F is not a contraction, since $F^n(\mathbb{K}(I)) \not\rightarrow \{I\}$ (see [Op] and [GKLOP, Theorem 10]).

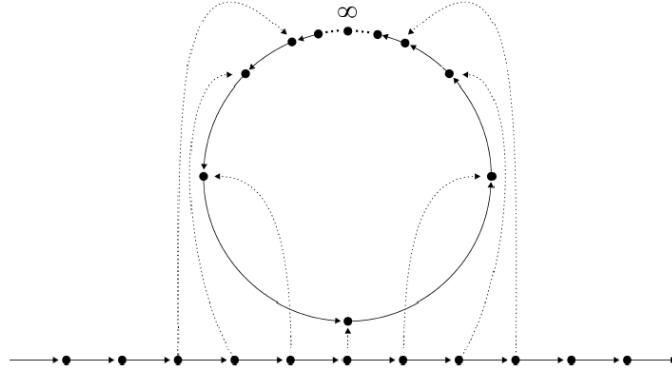


FIGURE 1.

REMARK 6. Since attractors and strict attractors are topological notions and we employed topological means to prove stability of attractors, the whole proof of stability could be conducted even for topological spaces with a trade off for lower comprehensibility.

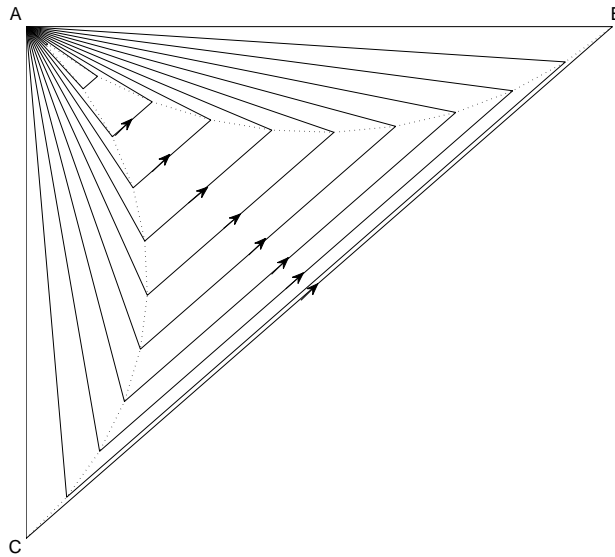


FIGURE 2.

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